

Geometric Methods for Improving the Upper Bounds on the Number of Rational Points on Algebraic Curves over Finite Fields

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with an Appendix by J-P.Serre

Abstract

Currently, the best upper bounds on the number of rational points on an absolutely irreducible, smooth, projective algebraic curve of genus g defined over a finite field \mathbb{F}_q come either from Serre's refinement of the Weil bound if the genus is small compared to q , or from Oesterlé's optimization of the explicit formulae method if the genus is large.

This paper presents three methods for improving these bounds. The arguments used are the indecomposability of the theta divisor of a curve, Galois descent, and Honda-Tate theory. Examples of improvements on the bounds include lowering them for a wide range of small genus when $q = 2^3, 2^5, 2^{13}, 3^3, 3^5, 5^3, 5^7$, and when $q = 2^{2s}$, $s > 1$. For large genera, isolated improvements are obtained for $q = 3, 8, 9$.

1 Introduction

This paper presents several methods and results for improving the upper bounds on the number of rational points on curves over finite fields. The first upper bound was discovered in the 1940s by André Weil as a direct result of proving the Riemann Hypothesis for curves. Weil showed that the number of rational points, N , on a smooth curve of genus g over the field \mathbb{F}_q satisfies the inequality

$$(W) \quad N \leq q + 1 + 2g\sqrt{q}.$$

For several decades, number theorists assumed that the Weil bound was optimal until 1973 when Stark [11] improved the bound by two in a particular case.

Dramatic improvements began in the 1980s, after Goppa discovered that curves over finite fields with many rational points could be used to construct efficient error-correcting codes. In 1981, Ihara found that equality in the Weil bound can only be achieved if the genus satisfies

$$g \leq \frac{\sqrt{q}(\sqrt{q} - 1)}{2}.$$

In [6], Serre proved a refined version of the Weil bound:

$$(SW) \quad N \leq q + 1 + gm, \quad m = [2\sqrt{q}],$$

where $[x]$ is the greatest integer part of x . The (SW) bound is the same as (W) if e is even. For small genus ($g \leq \frac{\sqrt{q}(\sqrt{q}-1)}{2}$), (SW) is the best upper bound known in many cases. For large genus ($g > \frac{\sqrt{q}(\sqrt{q}-1)}{2}$), Serre introduced the so-called explicit formulae method which Oesterlé optimized (see [7]). Since its discovery, the explicit formulae method has provided the best upper bounds in large genus.

Several improvements to (SW) in the small genus range are already known, notably Stark's improvement for $q = 13$, $g = 2$, (see [11]); Serre's generalization for all $g \geq 2$ and q of the form $q = x^2 + 1$ or $q = x^2 + x + 1$, (see [7]); Voloch's improvement for $g = 3$, $q \leq 25$, (see [7]); and the improvement when q is a square and the genus is in the range $\frac{(\sqrt{q}-1)^2}{4} < g < \frac{q-\sqrt{q}}{2}$ (see [1]).

The purpose of this paper is to present three geometric methods for further improving the bounds. The three methods are applications of Galois descent, Honda-Tate theory, and arguments on endomorphisms of the Jacobian of a curve. Examples of improvements obtained via these methods were announced in [5]. For a wide range of small genus we have improved the bounds by two when $q = 2^3, 2^5, 2^{13}, 3^3, 3^5, 5^3, 5^7$, and by one when $q = 2^{2s}$, $s > 1$. For large genus, we obtain isolated improvements for $q = 3, 2^3, 3^2$.

This paper is organized according to defect. A curve has *defect* k if it fails to meet (SW) by k . In Section 2, we review Serre's derivation of the list of possible zeta functions for curves of defect 0, 1, and 2. In Section 3, we use Galois descent to treat the defect 0 case for $q = 2^3, 2^5, 2^{13}, 3^3, 3^5, 5^3, 5^7$. In Section 4, we use Honda-Tate theory to treat the defect 2 case for $q = 2^{2s}$, $s > 1$. Finally, in Section 5, we generate lists of possible zeta functions for some higher defect cases to improve several bounds for $q = 3, 8, 9$. The largest defect we are able to treat is one case of defect 8.

The results of this paper explain why numerous construction attempts have failed to produce curves meeting the explicit formulae bounds in many cases. Furthermore, the findings presented here suggest that many more improvements on the bounds can be made by using a combination of these and other geometric methods.

Acknowledgements: I would like to thank J-P. Serre, James Milne, and René Schoof for their generous help and enthusiasm. In particular, I am grateful to J-P. Serre for suggesting revisions and for his appendix to improve the proof of Lemma 1. Also many thanks to René Schoof, Michael Bennett, and James McLaughlin for pointing out solutions to the diophantine equations in Section 3.1.

2 Defect k

NOTATION. Let $q = p^e$, with p prime, $e \geq 1$. By a curve over \mathbb{F}_q , we mean a smooth, projective, absolutely irreducible curve. For such a curve, C , let

$g = g(C)$ denote the genus, and $N = N(C)$ denote the number of rational points over \mathbb{F}_q .

In this section we investigate the possibilities for *defect* k curves.

Definition A curve C has *defect* k if $N(C) = q + 1 + gm - k$, $m = [2\sqrt{q}]$.

We explain here the idea used in [7] to generate the list of possible zeta functions for all (q, g) when $N = q + 1 + gm - k$. First define the set

$$F_k = \{t^d - a_1 t^{d-1} + \cdots + a_d \in \mathbb{Z}[t] \mid a_1 = d + k \text{ and all roots are real } > 0 \}.$$

Let F_k^{irred} be the subset of F_k consisting of irreducible polynomials. Then it follows from Siegel's theorem that F_k^{irred} is a finite set for $k \geq 0$. For a fixed d , the set of elements of F_k of degree d is also finite, and can be listed by taking all products of elements f_j of degree d_j in $F_{k_j}^{irred}$ such that $\sum k_j = k$ and $\sum d_j = d$. In [10], Smyth produced complete lists of the sets F_k^{irred} , for $k \leq 6$.

We say that a curve has *zeta function of type* (x_1, \dots, x_g) if $\{\alpha_i, \bar{\alpha}_i\}$ is the family of g conjugate pairs of eigenvalues of Frobenius acting on the Jacobian of the curve, and $x_i = -(\alpha_i + \bar{\alpha}_i)$, $i = 1, \dots, g$. The $m + 1 - x_i$ are totally positive algebraic integers, so if

$$\sum_{i=1}^g x_i = gm - k,$$

then

$$P(t) = \prod_{i=1}^g (t - (m + 1 - x_i)) \in F_k,$$

since $\deg P = g$, and $a_1 = g + k$.

The numerator of the zeta function of a curve is determined by $\{x_i\}$, so a list of possible zeta functions for curves of defect k and genus g can be imported from the lists in [10] for $k \leq 6$. For $k \leq 2$, the lists were contained in [7], and we recall them in Table 1 for convenience.

For $k = 3$ the table would have 25 entries.

For each pair (q, g) , an entry in Table 1 might not correspond to the zeta function of a curve for a number of possible reasons. Here are three such reasons from [9].

(2.1) Since the eigenvalues of Frobenius of a curve have absolute value \sqrt{q} , the x_i must satisfy $|x_i| \leq 2\sqrt{q}$. Any entry in the table not satisfying this condition for all i can be eliminated. Write $2\sqrt{q} = m + \{2\sqrt{q}\}$, where $\{x\}$ denotes the fractional part of x . Then for example the fourth entry for defect 2 is only possible if $\sqrt{3} - 1 \leq \{2\sqrt{q}\}$.

(2.2) The zeta function of the curve can be expressed in terms of the $\{x_i\}$ for each entry in the table. For example if the Jacobian of the curve is isogenous

Table 1: Possibilities for (x_1, \dots, x_g) for defect k , with genus restriction

k	(x_1, \dots, x_g)	g
0	(m, \dots, m)	
1	$(m, \dots, m, m-1)$	$g \geq 1$
	$(m, \dots, m, m + \frac{-1+\sqrt{5}}{2}, m + \frac{-1-\sqrt{5}}{2})$	$g \geq 2$
2	$(m, \dots, m, m-2)$	$g \geq 1$
	$(m, \dots, m, m-1, m-1)$	$g \geq 2$
	$(m, \dots, m, m + \sqrt{2} - 1, m - \sqrt{2} - 1)$	$g \geq 2$
	$(m, \dots, m, m + \sqrt{3} - 1, m - \sqrt{3} - 1)$	$g \geq 2$
	$(m, \dots, m, m-1, m + \frac{-1+\sqrt{5}}{2}, m + \frac{-1-\sqrt{5}}{2})$	$g \geq 3$
	$(m, \dots, m, m + \frac{-1+\sqrt{5}}{2}, m + \frac{-1-\sqrt{5}}{2}, m + \frac{-1+\sqrt{5}}{2}, m + \frac{-1-\sqrt{5}}{2})$	$g \geq 4$
	$(m, \dots, m, m + 1 - 4 \cos^2 \frac{\pi}{7}, m + 1 - 4 \cos^2 \frac{2\pi}{7}, m + 1 - 4 \cos^2 \frac{3\pi}{7})$	$g \geq 3$

over \mathbb{F}_q to a product of elliptic curves, then

$$L(t) = \prod_{i=1}^g (1 + x_i t + q t^2),$$

and the zeta function is

$$Z(t) = \frac{L(t)}{(1-t)(1-qt)}.$$

Re-writing in the form

$$1/Z(t) = \prod_{i=1}^{\infty} (1 - t^i)^{a_i},$$

we must have $a_i \geq 0$ for all i , since a_i is the number of places of degree i on the curve. So any entry which does not satisfy this condition can be eliminated.

(2.3) An entry does not correspond to a curve if the Jacobian admits a non-trivial decomposition into a product as a polarized abelian variety. This condition eliminates any entry for which the set $\{x_i\}$ can be partitioned into two non-empty subsets, I and J , such that each set is permuted by $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ and such that the difference between any element of I and any element of J is a unit. The proof of this fact can be found in [7] or as Lemma 1 in [3].

A combination of applications of these three reasons provides the following facts.

Proposition 1 *The (SW) bound can only be attained if*

$$g \leq \frac{q^2 - q}{m + m^2 - 2q}.$$

Proof: Suppose a curve of genus g attains (SW) so that $x_i = m$, for all i . The coefficients of the polynomial $(T + m)^g$ can be computed in two ways: as binomial coefficients or via Newton's relations between the elementary symmetric functions, $\{b_n\}$, and the power functions,

$$s_n = \sum_{i=1}^g (\alpha_i + \bar{\alpha}_i)^n.$$

Using the identity

$$b_2 = \frac{1}{2}(s_1^2 - s_2),$$

and equating the coefficients of the $g - 2$ term computed in the two ways yields:

$$\binom{g}{2} m^2 = \frac{1}{2}((gm)^2 - (q^2 + 1 - (q + 1 + gm + 2a_2) + 2gq)).$$

By reason (2.2), we must have $a_2 \geq 0$, so rearranging yields the desired inequality.

Remark 1 Note that Proposition 1 generalizes Ihara's result [2] that the Weil bound cannot be met unless $g \leq (q - \sqrt{q})/2$.

Proposition 2 *There are no defect 1 curves of genus $g > 2$.*

Proof: This fact was observed in [7], due to reason (2.3), since both entries for defect 1 curves can be suitably partitioned if $g > 2$. In the case $g = 2$, defect 1 is only possible if $\frac{\sqrt{5}-1}{2} \leq \{2\sqrt{q}\}$.

Proposition 3 *If e is even, then the only defect 2 curves with genus $g > 2$ have zeta function of type $(m, \dots, m, m - 2)$.*

Proof: Since $2\sqrt{q} = m$, it follows from reason (2.1) that the only possibilities are

$$(m, \dots, m, m - 2)$$

and

$$(m, \dots, m, m - 1, m - 1).$$

If $g > 2$, then $(m, \dots, m, m - 1, m - 1)$ is not possible by reason (2.3).

Proposition 4 *Let $g \geq 3$, $g \neq 4$. If q satisfies $\{2\sqrt{q}\} < \sqrt{3} - 1$, then defect 2 is only possible if*

$$g \leq \frac{q^2 - q - 2 + 4m}{m + m^2 - 2q}.$$

(If $g = 4$, then the same conclusion holds if q also satisfies $\{2\sqrt{q}\} < \frac{\sqrt{5}-1}{2}$).

Proof: If $g \geq 5$, then by reason (2.3), the only possibilities are

$$(m, \dots, m, m-2)$$

and

$$(m, \dots, m, m + \sqrt{3} - 1, m - \sqrt{3} - 1).$$

The second possibility is eliminated by reason (2.1) since $\{2\sqrt{q}\} < \sqrt{3} - 1$. The condition on the genus comes from a computation similar to the one in Proposition 1. Computing the coefficient of the $g-2$ term in

$$(T+m)^{g-1}(T+(m-2))$$

in two different ways and equating yields:

$$\frac{1}{2}(g-1)(gm^2 - 4m) = \frac{1}{2}((gm-2)^2 - (q^2 + 1 - (q-1 + gm + 2a_2) + 2gq)).$$

Since $a_2 \geq 0$ by reason (2.2), we obtain the stated restriction on the genus.

If $g = 3$, the last entry in Table 1 is eliminated by reason (2.1) since

$$\{2\sqrt{q}\} < \sqrt{3} - 1 < 1 - 4\cos^2 \frac{3\pi}{7}.$$

If $g = 4$, we must also assume that $\{2\sqrt{q}\} < \frac{\sqrt{5}-1}{2}$.

3 Galois Descent

When the genus satisfies the inequality of Proposition 1, we say that the genus is *small* (compared to q). For small genus, (SW) is often the best upper bound known. Here are some cases where we can improve it by 2.

Theorem 1 *The (SW) bound cannot be met in the following cases:*

$$\begin{aligned} q = 2^3, & \quad 4 \leq g, \\ q = 2^5, & \quad 3 \leq g, \\ q = 2^{13}, & \quad 4 \leq g, \\ q = 3^3, & \quad 3 \leq g, \\ q = 3^5, & \quad 4 \leq g, \\ q = 5^3, & \quad 4 \leq g, \\ q = 5^7, & \quad 7 \leq g. \end{aligned}$$

Remark 2 Note that Serre had already deduced the result in the case $q = 27$, $g = 3$, by using Hermitian modules. The result for the case $q = 243$, $g = 3$ also follows from that argument. The theorem does not extend to the case $q = 8$, $g = 3$, since the Klein curve has 24 rational points in that case.

Proof: Theorem 1 is proved by supposing that a curve meeting (SW) over \mathbb{F}_q does exist, and using Galois descent to produce an \mathbb{F}_p -structure for the curve which leads to a contradiction for the stated cases. The proof rests on the following descent lemma which was used in [7] to resolve the genus 2 case.

Lemma 1 *Let X be a curve over \mathbb{F}_q , $q = p^e$, p prime, e odd, of genus g , $g \geq 2$, with eigenvalues of Frobenius $\{\pi, \bar{\pi}\}$ repeated g times. If*

$$\pi = \sigma^e, \quad \text{with } \sigma \in \mathbb{Z}[\pi],$$

then X has an \mathbb{F}_p -structure with Frobenius endomorphism σ .

Proof of Lemma 1: The idea of the proof is as follows. All details are contained in the appendix. If the Jacobian of the curve descends to \mathbb{F}_p with its polarization, then the curve also descends by the precise version of the Torelli theorem which is stated in the first section of the appendix. In order to descend the Jacobian, it is necessary and sufficient ([8], Prop. 2, p.110) that σ factors as

$$\sigma = \phi \circ \theta,$$

where θ is the relative Frobenius map and ϕ is a biregular isomorphism. The fact that σ satisfies this condition if $\sigma \in \mathbb{Z}[\pi]$ with $\pi = \sigma^e$ and $q = p^e$ with p prime is shown in Theorem 6' of the appendix. To show that the principal polarization also descends, it is necessary and sufficient to show that

$$\sigma\sigma' = p,$$

where σ' is the endomorphism obtained from σ by applying the involution associated to the polarization. The fact that $\sigma\sigma' = p$ if $\sigma \in \mathbb{Z}[\pi]$ is shown in the corollary to Theorem 8 of the appendix. \square

To complete the proof of Theorem 1, we first establish that in the cases stated in the theorem, a curve meeting (SW) would satisfy the hypotheses of Lemma 1. If $N(X) = q + 1 + gm$, then the eigenvalues are $\{\pi, \bar{\pi}\}$, repeated g times, with

$$\pi = \frac{-m \pm \sqrt{m^2 - 4q}}{2}.$$

- $q = 2^3, \quad m = 5, \quad \pi = \frac{-5 - \sqrt{-7}}{2} = \sigma^3, \quad \sigma = \frac{1 + \sqrt{-7}}{2}, \quad \sigma = -\pi - 2.$
- $q = 2^5, \quad m = 11, \quad \pi = \frac{-11 + \sqrt{-7}}{2} = \sigma^5, \quad \sigma = \frac{-1 - \sqrt{-7}}{2}, \quad \sigma = -\pi - 6.$
- $q = 2^{13}, \quad m = 181, \quad \pi = \frac{-181 - \sqrt{-7}}{2} = \sigma^{13}, \quad \sigma = \frac{1 + \sqrt{-7}}{2}, \quad \sigma = -\pi - 90.$
- $q = 3^3, \quad m = 10, \quad \pi = -5 + \sqrt{-2} = \sigma^3, \quad \sigma = 1 + \sqrt{-2}, \quad \sigma = \pi + 6.$
- $q = 3^5, \quad m = 31, \quad \pi = \frac{-31 - \sqrt{-11}}{2} = \sigma^5, \quad \sigma = \frac{-1 - \sqrt{-11}}{2}, \quad \sigma = \pi + 15.$
- $q = 5^3, \quad m = 22, \quad \pi = \frac{-22 - \sqrt{-16}}{2} = \sigma^3, \quad \sigma = 1 + 2i, \quad \sigma = -\pi - 10.$

- $q = 5^7$, $m = 559$, $\pi = \frac{-559 + \sqrt{-19}}{2} = \sigma^7$, $\sigma = \frac{1 + \sqrt{-19}}{2}$, $\sigma = \pi + 280$.

Thus X has an \mathbb{F}_p -structure with Frobenius σ , so we examine the number of rational points over \mathbb{F}_p or an extension of \mathbb{F}_p ,

$$\#X(\mathbb{F}_{p^e}) = p^e + 1 - g\text{Tr}(\sigma^e).$$

- $q = 2^3$: Over \mathbb{F}_2 , $\#X(\mathbb{F}_2) = 2 + 1 - g$, which is impossible for $g \geq 4$. In addition, $\#X(\mathbb{F}_4) = 4 + 1 + 3g$, which is possible for $g = 3$, but not for $g = 2$ or $g \geq 4$. This gives another proof of the fact from [7] that (SW) cannot be met for $q = 8$, $g = 2$.
- $q = 2^5$: Over \mathbb{F}_8 , $\#X(\mathbb{F}_8) = 8 + 1 - 5g$, which is impossible for $g \geq 2$.
- $q = 2^{13}$: Over \mathbb{F}_2 , $\#X(\mathbb{F}_2) = 2 + 1 - g$, which is impossible for $g \geq 4$.
- $q = 3^3$: Over \mathbb{F}_3 , $\#X(\mathbb{F}_3) = 3 + 1 - 2g$, which is impossible for $g \geq 3$.
- $q = 3^5$: Over \mathbb{F}_{27} , $\#X(\mathbb{F}_{27}) = 27 + 1 - 8g$, which is impossible for $g \geq 4$.
- $q = 5^3$: Over \mathbb{F}_5 , $\#X(\mathbb{F}_5) = 5 + 1 - 2g$, which is impossible for $g \geq 4$.
- $q = 5^7$: Over \mathbb{F}_5 , $\#X(\mathbb{F}_5) = 5 + 1 - g$, which is impossible for $g \geq 7$.

This completes the proof of Theorem 1. \square

For fixed g and q , let $N_q(g)$ denote the maximum of $N(C)$ as C runs through all curves of genus g over \mathbb{F}_q .

Corollary 1 *If $q = 2^3, 2^{13}, 3^5, 5^3$ (resp. $q = 2^5, 3^3$), and $g \geq 4$ (resp. $g \geq 3$), then*

$$N_q(g) \leq q - 1 + gm.$$

Proof: (SW) cannot be met by Theorem 1, and Proposition 2 implies that defect 1 is impossible.

Example 1

$$N_8(4) \leq 27,$$

$$N_{32}(3) \leq 64,$$

$$N_{27}(3) \leq 56,$$

$$N_{27}(4) \leq 66.$$

3.1 A pair of Diophantine equations

Cases where Lemma 1 improves the upper bounds for the number of rational points on curves over a finite field \mathbb{F}_q correspond to integer solutions to a pair of diophantine equations:

$$(3.1) \quad x^2 + d = 4p,$$

$$(3.2) \quad y^2 + d = 4p^e,$$

where d is positive, e is odd, p is prime, and $q = p^e$. Provided that $d < 2y + 1$, we have $y = m = \lfloor \sqrt{4q} \rfloor$, and so $-d = m^2 - 4q$ as in the instances of Theorem 1. The correspondence is expressed by the following lemma.

Lemma 2 *A solution (x, y, d, p, e) to the pair of equations (3.1) and (3.2) with $3 < d < 2y + 1$, d square-free, corresponds to a pair of algebraic integers π and σ which satisfy the conditions of Lemma 1:*

$$\pi = \frac{-y \pm \sqrt{-d}}{2}, \quad \sigma = \frac{x \pm \sqrt{-d}}{2}, \quad \sigma^e = \pm\pi, \quad \sigma \in \mathbb{Z}[\pi], \quad \pi\bar{\pi} = q, \quad \sigma\bar{\sigma} = p.$$

Proof: Given a solution to (3.1) and (3.2), we must show that π and σ as defined in the statement satisfy the required properties. All properties are immediate except $\sigma^e = \pm\pi$. This follows from the fact that R , the ring of integers in $\mathbb{Q}(\pi)$, is a unique factorization domain with only the trivial units $\{\pm 1\}$. In fact, p splits in R as $p = \sigma\bar{\sigma}$, but p does not divide π . Since $p^e = \pi\bar{\pi}$ we must have π or $\bar{\pi}$ associated to σ^e . \square

Example 2 Note that if $q = 7^3$, we have the following solution to (3.1) and (3.2): $x = 5$, $y = 37$, $d = 3$. This solution does not correspond to a pair satisfying the conditions of Lemma 1 however, since there are non-trivial units in the ring of integers of $\mathbb{Q}(\sqrt{-3})$. Let $u = \frac{1+\sqrt{-3}}{2}$. Then if $\pi = \frac{-37-\sqrt{-3}}{2}$ and $\sigma = \frac{-5+\sqrt{-3}}{2}$, we have $\pi = u \cdot \sigma^3$.

In [7], Serre deduced that (SW) cannot be met when q is of the form $q = x^2 + 1$ or $x^2 + x + 1$ and the genus is at least 2. Theorem 1 was discovered as a result of trying to extend this theorem to prime powers of the form $q = x^2 + x + 2$. For q of the form $q = x^2 + x + 2$, we must have $p = 2$, and there are only 5 such q . They correspond to the famous solutions to (3.1) and (3.2) when $d = 7$ and $e = 1, 2, 3, 5, 13$, referred to in this case as the Ramanujan-Nagell equations.

In addition to the solutions to (3.1) and (3.2) listed in Theorem 1, there is a family of solutions when $e = 3$ which was pointed out by René Schoof and Michael Bennett. For any integer k such that $p = k^2 + 1$ is prime, we have a solution of the form

$$x = k, \quad y = k(2k^2 + 3), \quad d = 3k^2 + 4, \quad e = 3.$$

For $k = 1$ and $k = 2$, the solutions correspond to the first and the second-to-last instances of Theorem 1. This (conjecturally) infinite family of solutions improves the upper bounds for the number of rational points on curves for each of the corresponding fields, due to Lemma 2.

4 Honda-Tate Theory

Since the Weil and Serre bounds coincide when q is a square, we can consider the defect from the Weil bound. For $q = 2^{2s}$, $s > 1$ and g small in a certain range, we can improve the bounds due to the following theorem.

Theorem 2 *If $q = 2^{2s}$, $s > 1$, and $g > 2$, then there are no defect 2 curves.*

Proof: The proof of Theorem 2 relies on Honda-Tate theory. By Proposition 3, for q a square and $g > 2$, the only possibility for a defect 2 curve is one with its Jacobian isogenous to the product of elliptic curves:

$$E_m \times \cdots \times E_m \times E_{m-2},$$

where E_m is an elliptic curve with $\text{Tr}(\text{Frobenius}) = -m$. By Honda-Tate theory, when $q = 2^{2s}$, the only possible values for the trace of an elliptic curve which are divisible by the characteristic are (see [12], p.536)

$$\{0, \pm\sqrt{q}, \pm 2\sqrt{q}\}.$$

If $s > 1$, then $m - 2 = 2\sqrt{q} - 2$ is not on this list, so such an abelian variety is impossible.

Corollary 2 *If $q = 2^{2s}$, $s > 1$, and $\frac{(\sqrt{q}-1)^2}{4} < g < \frac{q-\sqrt{q}}{2}$, then*

$$N_q(g) \leq q - 2 + gm.$$

Proof: Due to a result of Fuhrmann and Torres [1] when q is a square, there are no defect 0 curves for any g in the interval

$$\frac{(\sqrt{q}-1)^2}{4} < g < \frac{q-\sqrt{q}}{2}.$$

By Proposition 2, there are no defect 1 curves for $g > 2$. By Theorem 2, there are no defect 2 curves for $g > 2$.

Example 3 Theorem 2 leads to the following improvements on the bounds:

$$N_{16}(4) \leq 46,$$

$$N_{16}(5) \leq 54,$$

$$N_{64}(g) \leq 62 + 16g, \quad \text{for } 13 \leq g \leq 27.$$

5 Zeta Functions

When the genus is large, the explicit formulae bounds force the maximal curves to have defect bigger than 2. In this case, we proceed by using reasons #1,2,3 from Section 2 directly to generate lists of possible zeta functions. In the following theorem we give several cases where the explicit formulae bounds can be improved by showing that the lists are empty.

Theorem 3 *The optimal form of the explicit formulae bounds cannot be met in the following cases:*

$$\begin{aligned} q = 3, \quad g = 5, \quad N = 14 \text{ (defect 5)} \\ q = 3, \quad g = 7, \quad N = 17 \text{ (defect 8)} \\ q = 9, \quad g = 5, \quad N = 36 \text{ (defect 4)} \\ q = 8, \quad g = 6, \quad N = 36. \text{ (defect 3)} \end{aligned}$$

Proof: The first two cases were proved in [3] and [4] respectively.

For $q = 9, g = 5$, the Weil and Serre bounds give $N \leq 40$. The optimal form of the explicit formulae bounds gives $N \leq 36$, which is defect 4. Applying the algorithm from [4], we find that the only possibility is

$$(m, m-1, m-1, m-1, m-1),$$

which is impossible by reason #3.

For $q = 8, g = 6$, the optimal form of the explicit formulae bounds again gives $N \leq 36$, which is defect 3. The possibilities are

$$\begin{aligned} (m, m, m, m, m-1, m-2), \\ (m, m, m, m-1, m-1, m-1), \\ (m, m, m-1, m-1, m-\frac{1+\sqrt{5}}{2}, m-\frac{1-\sqrt{5}}{2}), \end{aligned}$$

all three of which are impossible by reason #3.

Appendice

chère Kristin,

A propos de la descente du corps de base pour les courbes et leurs jacobiniennes:

1. Le théorème de Torelli

Soit k un corps. Par une “courbe” sur k j’entends une courbe projective, lisse, absolument irréductible. Si X est une telle courbe, son genre $g(X)$ sera noté g . On suppose $g > 1$. On note $\text{Jac } X$ la jacobienne de X munie de sa polarisation naturelle a , qui est de degré 1. Si X' est une autre courbe sur k , tout isomorphisme $f : X \rightarrow X'$ définit par transport de structure un isomorphisme $f_J : (J, a) \rightarrow (J', a')$, où (J', a') est la jacobienne de X' . Le **théorème de Torelli** dit que l’on obtient ainsi “presque” tous les isomorphismes $(J, a) \rightarrow (J', a')$. De façon plus précise:

Théorème 1 *Supposons X hyperelliptique. Pour tout isomorphisme de variétés abéliennes polarisées*

$$F : (J, a) \rightarrow (J', a'),$$

il existe un isomorphisme $f : X \rightarrow X'$ et un seul tel que $F = f_J$.

Théorème 2 *Supposons X non hyperelliptique. Alors, pour tout isomorphisme $F : (J, a) \rightarrow (J', a')$, il existe un isomorphisme $f : X \rightarrow X'$ et un entier e égal à ± 1 tel que $F = e \cdot f_J$. De plus, le couple (f, e) est déterminé par F de façon unique.*

(Noter que X est hyperelliptique si et seulement si il existe un automorphisme s de X tel que $s_J = -1$.)

Les ths.1 et 2 constituent ce que j'ai envie d'appeler la "forme précise" du théorème de Torelli (la forme imprécise consistant à dire seulement que X et X' sont isomorphes). Je ne crois pas que la "forme précise" se trouve explicitement dans la littérature. Toutefois:

—Lorsque k est algébriquement clos, c'est essentiellement l'énoncé démontré par Weil (*Oe.* II, [1957a]), à cela près que Weil choisit un plongement de X dans sa jacobienne, ce qui introduit des translations qui n'ont rien à voir avec la question. La démonstration du théorème de Torelli due à Andreotti (et reproduite par exemple dans Albarello-Cornalba-Griffiths-Harris, *Grundlehren* 267) ne donne que la forme imprécise.

—Le cas d'un corps parfait résulte du cas algébriquement clos par descente galoisienne standard (grâce à l'unicité de f ou (f, e)). Le cas d'un corps imparfait résulte de celui d'un corps parfait : en effet, si k_1 est une extension radicielle de k , tout isomorphisme de X/k_1 sur X'/k_1 est "défini sur k ", i.e. provient d'un isomorphisme de X sur X' (utiliser le fait que le schéma $\text{Isom}(X, X')$ est étale). D'ailleurs, dans la suite, le cas d'un corps parfait nous suffira.

2. Un corollaire du théorème de Torelli

C'est l'énoncé suivant, qui résulte immédiatement des ths.1 et 2:

Théorème 3 *On a*

$$\text{Aut}(J, a) = \begin{cases} \text{Aut} X & \text{si } X \text{ est hyperelliptique} \\ \{\pm 1\} \times \text{Aut} X & \text{si } X \text{ n'est pas hyperelliptique.} \end{cases}$$

Corollaire *Supposons que le groupe fini $\text{Aut}(J, a)$ contienne un élément s tel que $s^n = -1$, avec n pair. Alors X est hyperelliptique.*

En effet l'existence d'un tel s est incompatible avec la décomposition $\text{Aut}(J, a) = \{\pm 1\} \times \text{Aut} X$.

Cet énoncé peut être utile pour montrer que certaines courbes, construites par la méthode des modules hermitiens, sont hyperelliptiques.

3. Descente du corps de base

On se donne une extension galoisienne finie k_1/k , de groupe de Galois G . Pour éviter des indices trop abondants, on note X_1, J_1, \dots une courbe sur k_1 , sa jacobienne, etc. On se donne une k -structure sur J_1 , compatible avec la polarisation; cela revient à se donner un couple (J, a) , où J est une variété

abélienne sur k , munie d'une polarisation a définie sur k , et à se donner un isomorphisme de (J_1, a_1) avec $(J, a)/k_1$. On veut passer de la jacobienne à la courbe.

Théorème 4 *Supposons X_1 hyperelliptique. Il existe alors une k -structure unique sur X_1 , compatible avec sa k_1 -structure, et dont la jacobienne est (J, a) .*

Cela résulte du th.1, par descente à la Weil. De façon plus précise, si s est un élément donné de G , la k -structure (J, a) donne un isomorphisme de (J_1, a_1) sur son s -transformé $(J_1, a_1)^s$; d'où par le th.1 un isomorphisme $f_s : X_1 \rightarrow (X_1)^s$. Ces isomorphismes satisfont à la condition de cocycle usuelle. D'où la structure cherchée.

Le cas non hyperelliptique est analogue, mais plus amusant:

Théorème 5 *Supposons X_1 non hyperelliptique. Il existe alors une k -structure X sur X_1 , compatible avec sa k_1 -structure, et un homomorphisme $\epsilon : G \rightarrow \{\pm 1\}$, tel que la jacobienne de X soit isomorphe à la ϵ -tordue $(J, a)_\epsilon$ de (J, a) .*

(Par “ ϵ -tordue” j'entends la variété déduite de (J, a) par torsion galoisienne relativement à $\epsilon : G \rightarrow \{\pm 1\} \subset \text{Aut}(J, a)$.)

La démonstration est la même que celle du th 4. Pour chaque $s \in G$ on a (cf. th.2) un isomorphisme $f_s : X_1 \rightarrow (X_1)^s$ ainsi qu'un signe $e_s = \pm 1$. On définit alors ϵ par $\epsilon(s) = e_s$.

Remarque. On pourrait sûrement déduire les ths. 4 et 5 d'un énoncé portant sur le morphisme de “champs”:
“champ de courbes” \rightarrow “champ de variétés abéliennes à polarisation principale”.

4. Corps finis : variétés abéliennes

On va s'intéresser maintenant au cas où k est un corps fini à q éléments et k_1 une extension finie à q_1 éléments, avec $q_1 = q^r$, $r > 1$. On se donne une courbe X_1 sur k_1 , et l'on désire “descendre” son corps de définition à k , comme ci-dessus. Cela va se faire en trois étapes:

descente pour les variétés abéliennes;

descente pour les variétés abéliennes polarisées;

descente pour les courbes.

Occupons-nous du premier cas, i.e. de celui des variétés abéliennes. On se donne une variété abélienne A_1 sur k_1 . Notons π_1 son endomorphisme de Frobenius. Une k -structure sur A_1 est définie par son endomorphisme de Frobenius $\pi \in \text{End}(A_1)$. Les conditions que π doit satisfaire sont les suivantes:

Théorème 6 *Pour que π définisse sur A_1 une k -structure compatible avec sa k_1 -structure, il faut et il suffit que:*

- a) $\pi_1 = \pi^r$, où $r = [k_1 : k]$;
- b) π est nul sur le noyau N_q de l'homomorphisme de Frobenius absolu

$$F_q : A_1 \rightarrow (A_1)^{(q)}.$$

(Une façon équivalente de formuler b) est de dire que, pour toute fonction rationnelle h sur A_1 , la fonction $h \circ \pi$ est la puissance q -ième d'une fonction rationnelle.)

La nécessité de ces conditions est immédiate. La suffisance résulte par exemple de la prop.2 de [8], Chap.VI § 1, p.110: d'après cette proposition, on doit vérifier que π est de la forme $\sigma \circ F_q$, où σ est un isomorphisme de $(A_1)^{(q)}$ sur A_1 . Or b) entraîne que π se factorise en $\sigma \circ F_q$, où σ est un homomorphisme de $(A_1)^{(q)}$ dans A_1 . Comme les degrés de π et de F_q sont tous deux égaux à $q^{\dim(A)}$, on voit que σ est de degré 1, i.e. que c'est un isomorphisme.

Remarque. On peut donner des exemples où a) est vérifiée, mais pas b).

Toutefois:

Théorème 6' *Supposons que la condition a) du th.6 soit satisfaite. Faisons les hypothèses suivantes :*

c) q est égal à la caractéristique p du corps k ;

d) il existe un polynôme $P(X)$ à coefficients entiers tel que π soit égal à $P(\pi_1)$.

Alors la condition b) du th.6 est satisfaite.

Ecrivons π sous la forme $a_0 + a_1\pi_1 + \dots + a_n\pi_1^n$, avec $a_i \in \mathbb{Z}$. L'application tangente à π_1 est nulle. Il en résulte que l'application tangente à π est l'homothétie de rapport a_0 . D'après a) la puissance r -ième de cette application est 0. Il en résulte que a_0 est divisible par p , d'où le fait que l'application tangente à π est 0. Or cela signifie que π s'annule sur N_p . La condition b) est donc satisfaite.

5. Corps finis : variétés abéliennes polarisées

On conserve les hypothèses du §4, et l'on suppose en outre que A_1 est munie d'une polarisation a_1 . On se donne $\pi \in \text{End}(A_1)$ satisfaisant aux conditions du th.6, donc définissant sur A_1 une k -structure. Soit A la variété abélienne ainsi obtenue. On désire donner des conditions permettant d'affirmer que a_1 est définie sur k , i.e. provient d'une polarisation a de A .

Je rappelle qu'une polarisation d'une variété abélienne définit une involution de l'algèbre $R_{\mathbb{Q}} = \mathbb{Q} \otimes R$, où $R = \text{End}(A)$ (cette involution laisse stable R lorsque la polarisation est degré 1, ce qui est le cas qui nous intéresse le plus). Je noterai $x \mapsto x'$ l'involution définie par la polarisation a_1 . En particulier, π' est défini.

Théorème 7 *Pour que la polarisation a_1 soit rationnelle sur k (pour la k -structure définie par π), il faut et il suffit que l'on ait*

e) $\pi\pi' = q$.

Notons V le "Verschiebung" de A , i.e. l'unique endomorphisme de A tel que $\pi V = q$. La condition e) ci-dessus équivaut à:

e') $\pi' = V$.

(Rappel de notations: si C est une variété abélienne, je note C^* sa duale; de même, si $h : B \rightarrow C$ est un homomorphisme, je note h^* l'homomorphisme correspondant ("transposé", "adjoint", ...) de C^* dans B^* . La polarisation $a_1 : A \rightarrow A^*$ est hermitienne: on a $a_1^* = a_1$. L'involution associée $x \mapsto x'$ de $\mathbb{Q} \otimes R$ est caractérisée par la formule $a_1.x' = x^*.a_1$.)

Il est bien connu que l'endomorphisme de Frobenius de A^* est égal à V^* . Le morphisme $a_1 : A \rightarrow A^*$ est rationnel sur K si et seulement si il commute au Frobenius, i.e. si et seulement si on a $a_1\pi = V^*a_1$. En comparant à l'équation $a_1V' = V^*a_1$, on voit que cela revient à $\pi = V'$, i.e. à $\pi' = V$. D'où le théorème.

Les conditions $e)$ et $e')$ peuvent être remplacées par une condition plus simple:

Théorème 8 *La condition $e)$ équivaut à :*
 $e'') \pi$ et π' commutent.

Il est clair que $e) \Rightarrow e'')$. Pour prouver la réciproque, il est commode d'utiliser l'algèbre $S = \mathbf{R} \otimes \text{End}(A)$, et la sous-algèbre T de S engendrée par π et π' . Vu $e'')$, cette algèbre est commutative, et stable par l'involution $x \mapsto x'$. De plus, si l'on pose $v = \pi/q^{1/2}$, et $z = vv'$, on a $z^r = \pi^r \pi'^r / q^r = (\pi_1 \pi'_1) / q^r = 1$. Or l'algèbre S (et donc aussi l'algèbre T) peut être munie (voir Mumford) d'une forme linéaire réelle t telle que $t(yy') > 0$ pour tout $y \neq 0$. Comme T est commutative, il en résulte que T se décompose en produit de corps isomorphes à \mathbb{R} ou \mathbf{C} , l'involution étant la conjugaison complexe. La formule $z = vv'$ montre que, dans chacun de ces corps, z est réel > 0 . Comme d'autre part c'est une racine de l'unité, on a $z = 1$, ce qui équivaut à $\pi\pi' = q$.

Corollaire *La condition $d)$ du th.6' entraîne la condition $e)$ du th.7.*

En effet, si π est un polynôme en π_1 , π' est un polynôme en π'_1 . Or π_1 et π'_1 commutent (puisque a_1 est définie sur k_1). Donc π et π' commutent, et l'on peut appliquer le th.8.

6. Corps finis: courbes

On se donne une courbe X_1 sur k_1 , et l'on note π_1 l'endomorphisme de Frobenius de sa jacobienne J_1 . On se donne $\pi \in \text{End}(J_1)$, avec $\pi^r = \pi_1$ et l'on cherche à mettre sur X_1 une k -structure telle que l'endomorphisme de Frobenius correspondant soit π . On suppose que π satisfait aux conditions $b)$ et $e)$ des ths. 6 et 7. Alors:

Théorème 9 *Si X_1 est hyperelliptique, ou si r est impair, il existe sur X_1 une k -structure dont le Frobenius est π .*

Sinon, il existe un signe $\epsilon = \pm 1$ et une k -structure sur X_1 dont le Frobenius est $\epsilon\pi$.

Vu les ths.6 et 7, il existe une k -structure sur J_1 , compatible avec sa k_1 -structure et sa polarisation, pour laquelle le Frobenius est π . Si X_1 est hyperelliptique, le th.4 donne l'existence de la k -structure cherchée sur X_1 . Si X_1 n'est pas hyperelliptique, le th.5 donne le même résultat, à cela près que la k -structure de J_1 doit être tordue par un caractère quadratique de $\text{Gal}(k_1/k)$, qui est un groupe cyclique d'ordre r . Si r est impair, un tel caractère est trivial; aucune torsion n'est donc nécessaire. Si r est pair, il se peut que ce caractère soit l'unique caractère non trivial; or l'effet d'une telle torsion est de remplacer π par son opposé. D'où le résultat cherché.

Remarque. Dans le cas qui vous intéresse, on a $q = p$, et π s'écrit comme polynôme en π_1 à coefficients entiers. Les conditions b) et c) sont alors satisfaites, et le th.9 s'applique.

C'est ce que l'on voulait.

Bien à vous

J.-P. Serre

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